

Stirling's formula derived simply

Joseph B. Keller

Departments of Mathematics and Mechanical Engineering

Stanford University, Stanford, CA 94305-2125

email: keller@math.stanford.edu

and

Jean-Marc Vanden-Broeck

Department of Mathematics

University College London

London WC1E 6BT, United Kingdom

email:j.vanden-broeck@ucl.ac.uk

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Abstract

Stirling's formula, the asymptotic expansion of $n!$ for n large, or of $\Gamma(z)$ for $z \rightarrow \infty$, is derived directly from the recursion equation $\Gamma(z+1) = z\Gamma(z)$ and the normalization condition $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Stirling's formula is an asymptotic expansion of the gamma function $\Gamma(z)$, valid as $z \rightarrow \infty$ with $|\arg z| < \pi$. Since $\Gamma(z)$ is the analytic extension of $(n-1)!$, it satisfies the recursion equation

$$\Gamma(z+1) = z\Gamma(z). \quad (1)$$

We shall derive Stirling's formula from (1) and the condition

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad (2)$$

which makes the solution unique.

Taking the logarithm of each side of (1), and writing $f(z) = \log \Gamma(z)$, yields

$$f(z+1) - f(z) = \log z. \quad (3)$$

Replacing $f(z+1)$ in (3) by its Taylor series around z leads to

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(z)}{n!} = \log z. \quad (4)$$

We integrate (4), introducing a constant of integration $\log C$, and we call the right side $f_0(z)$:

$$\sum_{n=1}^{\infty} \frac{f^{(n-1)}(z)}{n!} = z \log z - z + \log C \equiv f_0(z). \quad (5)$$

Keeping the undifferentiated term with $n=1$ on the left, and moving the other terms to the right, gives

$$f(z) = f_0(z) - \sum_{n=2}^{\infty} \frac{f^{(n-1)}(z)}{n!}. \quad (6)$$

Equation (6) is a differential equation for f of infinite order, which we solve by successive substitution. First we approximate f by f_0 . Then we use f_0 in place of f in the sum to get a second approximation to f . Then we substitute that for f in the sum, and so on. This will lead to an expansion for f of the form

$$f(z) \sim \sum_{j=0}^{\infty} c_j f_0^{(j)}(z). \quad (7)$$

We expect (7) to be an asymptotic expansion of $f(z)$, valid as $z \rightarrow \infty$. Therefore we have used the sign \sim of asymptotic equality.

To determine the coefficients c_j in (7) we substitute (7) into (6) and equate coefficients of $f_0^{(j)}$. This yields

$$c_0 = 1, \quad c_j = - \sum_{k=0}^{j-1} \frac{c_k}{(j-k+1)!}, \quad j \geq 1. \quad (8)$$

Equation (8) determines the c_j successively, starting with $c_0 = 1$, $c_1 = -1/2$, etc. To solve it in general we set $c_j = B_j/j!$ in (8). The resulting recursion equation for B_j is exactly that satisfied by the Bernoulli numbers [eq. (6.79), p. 270 of Graham, Knuth and Patashnik [1]]. Thus B_j is the j -th Bernoulli number.

The expansion (7) of f involves the derivatives of f_0 . Since $f_0^{(1)} = \log z$ it follows that $f_0^{(j)} = (-1)^j(j-2)!/z^{j-1}$ for $j \geq 2$. For $j \geq 2$ with j odd, $B_j = 0$. Therefore by setting $j = 2n$ for $j \geq 2$, and using the definition $f(z) = \log \Gamma(z)$, we can write (7) as follows:

$$\log \Gamma(z) \sim \left(z - \frac{1}{2}\right) \log z - z + \log C + \sum_{n=1}^{\infty} \frac{(2n-2)! B_{2n}}{(2n)! z^{2n-1}}. \quad (9)$$

Exponentiating both sides of (9) gives

$$\Gamma(z) \sim C z^{z-\frac{1}{2}} e^{-z} \exp \left[\sum_{n=1}^{\infty} \frac{(2n-2)! B_{2n}}{(2n)! z^{2n-1}} \right]. \quad (10)$$

To find C we first prove that

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\Gamma(n+1)}. \quad (11)$$

For $n = 0$ this reduces to (2), so it is true. It follows for $z = n$ by induction on n using (1).

When we use the leading term in (10) for $\Gamma(n)$ in (11), we find that

$$C = \sqrt{2\pi}. \quad (12)$$

This completes the determination of the expansions (9) and (10) of $\log \Gamma$ and of Γ . With $C = \sqrt{2\pi}$, (9) agrees with eq. (6.1.40) of Abramowitz and Stegun [2]. When the exponential

in (10) is expanded in a power series, it agrees with eq. (6.1.37) of Abramowitz and Stegun [2], which is Stirling's formula.

References

- [1] Graham, R., Knuth, D. and Patashnik, O., *Concrete Mathematics*, Addison-Wesley, New York, 1989.
- [2] Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, Dover, New York, 1970.